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## *ON UNSYMMETRICAL ADJUSTMENTS, AND THEIR LIMITS.*

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(*Continued from page* 148.)

LET us now consider the simplest case under (28), putting $n=1$. If, as usually happens, $b_2$ is positive, (22) shows that $a$ is negative. We will take

$$a = -2h^2, \qquad \therefore\ h^2 = \frac{1}{2b_2(k+1)(dx)^2}. \tag{30}$$

Integrating and applying the condition (23), and also writing $Y = y \div dx$, we find that the curve is the probability curve

$$y = \frac{hdx}{\sqrt{\pi}} e^{-h^2x^2}, \qquad \therefore\ Y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}. \tag{31}$$

or, since $x = idx$, and putting $(hdx)^2 = g$,

$$g = \frac{1}{2b_2(k+1)}, \qquad y = \sqrt{\left(\frac{g}{\pi}\right)}\, e^{-gi^2}. \tag{32}$$

Although these formulas are strictly true only when $k$ and $i$ are infinities of the second and first orders, respectively, so that $y$ is infinitesimal, yet they give approximately true results when these numbers are finite. Take for example eight applications of

$$u'_0 = \tfrac{1}{6}(u_{-1} + 3u_0 + 2u_1).$$

Here we have $\lambda_0 = \frac{3}{6}$, $\lambda_1 = \frac{2}{6}$, $\lambda_{-1} = \frac{1}{6}$, $b_0 = 1$, $k = 8$.

The place of the centre of parallel forces is

$$\frac{1(-1) + 3\times 0 + 2\times 1}{1+3+2} = \tfrac{1}{6},$$

or one sixth of the way from $u_0$ to $u_1$. The value of $b_2$ estimated with reference to this centre as an origin is,

$$\frac{1}{6}\left\{1\left(-\frac{7}{6}\right)^2+3\left(-\frac{1}{6}\right)^2+2\left(\frac{5}{6}\right)^2\right\}=\frac{17}{36}.$$

Formula (32) then gives

$$g=\frac{2}{17}, \qquad y=\sqrt{\left(\frac{2}{17\pi}\right)}\,e^{-\frac{2}{17}i^2},$$

$$\log y = \bar{1}.28671 - .051093\,i^2.$$

The origin or vertex of the curve will be placed at an interval of $k \times \frac{1}{6} = \frac{4}{3}$ to the right of $u_0$, so that the resultant coefficient of $u_0$ is found by putting $i = -\frac{4}{3}$, and proceeding from this at intervals of unity we get the other coefficients $y$ in Table I. They do not differ greatly from the true coefficients $l$ of the resultant formula, which are also shown in the table, as given by expanding the polynomial

$$\left(\frac{1+3z+2z^2}{6}\right)^8,$$

and the approximation would be closer if $k$ were greater.

TABLE I.

| $i$ | $y$ | $l$ | $i$ | $y$ | $l$ | $i$ | $y$ | $l$ |
|---|---|---|---|---|---|---|---|---|
| $-\frac{19}{3}$ | .002 | .001 | $-\frac{7}{3}$ | .102 | .099 | $\frac{8}{3}$ | .084 | .084 |
| $-\frac{16}{3}$ | .007 | .005 | $-\frac{4}{3}$ | .157 | .158 | $\frac{11}{3}$ | .040 | .035 |
| $-\frac{13}{3}$ | .021 | .018 | $-\frac{1}{3}$ | .191 | .198 | $\frac{14}{3}$ | .015 | .010 |
| $-\frac{10}{3}$ | .052 | .049 | $\frac{2}{3}$ | .184 | .194 | $\frac{17}{3}$ | .004 | .002 |
| | | | $\frac{5}{3}$ | .140 | .147 | | | |

We pass now to the second case under (28) where $n = 2$, and the equation and its conditions are

$$\frac{d^2y}{dx^2} = axy, \qquad \frac{1}{dx}\int_{-\infty}^{\infty} y\,dx = 1, \qquad \int_{-\infty}^{\infty} x^2 y\,dx = 0. \qquad (33)$$

To integrate this by series, using the method of indeterminate coefficients, we assume

$$y = A+Bx+Cx^2+Dx^3+\&c.,$$

$$\therefore \begin{cases} \dfrac{d^2y}{dx^2} = 1.2C+2.3Dx+3.4Ex^2+4.5Fx^3+\&c., \\ axy = Aax+Bax^2+Cax^3+\&c., \end{cases}$$

these last two being equal, so that

$$1.2C+(2.3D-Aa)x+(3.4E-Ba)x^2+(4.5F-Ca)x^3+\&c. = 0.$$

Coefficients of each power of $x$ being separately zero,

$$C=0, \qquad D=\frac{Aa}{2.3}, \qquad E=\frac{Ba}{3.4}, \qquad F=0,$$

$$G=\frac{Da}{5.6}=\frac{Aa^2}{2.3.5.6}, \qquad H=\frac{Ea}{6.7}=\frac{Ba^2}{3.4.6.7}, \text{\&c.},$$

and the equation of the curve is

$$y = A\left\{1+\frac{ax^3}{2.3}+\frac{(ax^3)^2}{2.3.5.6}+\frac{(ax^3)^3}{2.3.5.6.8.9}+\text{\&c.},\right\}$$
$$+Bx\left\{1+\frac{ax^3}{3.4}+\frac{(ax^3)^2}{3.4.6.7}+\frac{(ax^3)^3}{3.4.6.7.9.10}+\text{\&c.}\right\}. \qquad (34)$$

We may write it thus,

$$y=A[1+H_1(ax^3)+H_2(ax^3)^2+\text{\&c.}]+Bx[1+K_1(ax^3)+K_2(ax^3)^2+\text{\&c.}],$$

and tabulate the logarithms of $H_n$ and $K_n$ for the first 14 terms of each series, as below.

TABLE II.

| $n$ | $\log H_n$ | $\log K_n$ | $n$ | $\log H_n$ | $\log K_n$ |
|---|---|---|---|---|---|
| 1 | $\bar{1}.2218487$ | $\bar{2}.9208188$ | 8 | $\overline{17}.5936919$ | $\overline{18}.7931718$ |
| 2 | $\bar{3}.7447274$ | $\bar{3}.2975695$ | 9 | $\overline{20}.7473548$ | $\overline{21}.9146500$ |
| 3 | $\bar{5}.8873949$ | $\bar{5}.3433270$ | 10 | $\overline{23}.8078355$ | $\overline{24}.9461671$ |
| 4 | $\bar{7}.7668210$ | $\bar{7}.1502024$ | 11 | $\overline{26}.7841716$ | $\overline{27}.8961742$ |
| 5 | $\bar{9}.4446017$ | $\overline{10}.7699912$ | 12 | $\overline{29}.6838011$ | $\overline{30}.7716700$ |
| 6 | $\overline{12}.9588803$ | $\overline{12}.2359651$ | 13 | $\overline{32}.5129529$ | $\overline{33}.5785454$ |
| 7 | $\overline{14}.3356310$ | $\overline{15}.5713231$ | 14 | $\overline{35}.2769198$ | $\overline{36}.3218277$ |

The two conditions in (33) suffice to determine the two constants $A$ and $B$. In making the integrations we consider as before (ANALYST, September, 1878, p. 135), that $y$ will be so small as to be insensible when $x$ equals or exceeds a moderately large finite value $x_1$, so that it is not necessary to integrate up to infinity. We have then

$$\int_{-x_1}^{x_1} y\,dx = 2Ax_1[1+\tfrac{1}{7}H_2(ax_1^3)^2+\tfrac{1}{13}H_4(ax_1^3)^4+\tfrac{1}{19}H_6(ax_1^3)^6+\text{\&c.}]$$
$$+\ 2Bx_1^2[\quad \tfrac{1}{5}K_1(ax_1^3)+\tfrac{1}{11}K_3(ax_1^3)^3+\tfrac{1}{17}K_5(ax_1^3)^5+\text{\&c.}],$$

$$\int_{-x_1}^{x_1} x^2y\,dx = 2Ax_1^3[\tfrac{1}{3}+\tfrac{1}{9}H_2(ax_1^3)^2+\tfrac{1}{15}H_4(ax_1^3)^4+\text{\&c.}]$$
$$+\ 2Bx_1^4[\quad \tfrac{1}{7}K_1(ax_1^3)+\tfrac{1}{13}K_3(ax_1^3)^3+\text{\&c.}].$$

Denoting the sums of the terms in these four series by $S_1$, $S_2$, $S_3$, $S_4$, the conditions (33) give

$$2Ax_1S_1+2Bx_1^2S_2=dx, \qquad AS_3+Bx_1S_4=0,$$

from which we get

$$A=\frac{S_4\,dx}{2x_1(S_1S_4-S_2S_3)}, \qquad \frac{B}{A}=-\frac{S_3}{x_1S_4}.$$

Putting $ax_1^3 = c$, we have $x_1 = \sqrt[3]{(c \div a)}$, and therefore

$$A = \frac{S_4 dx}{2(S_1 S_4 - S_2 S_3)} \left(\frac{a}{c}\right)^{\frac{1}{3}}, \qquad \frac{B}{A} = -\frac{S_3}{S_4}\left(\frac{a}{c}\right)^{\frac{1}{3}}. \tag{35}$$

Since $S_1$, $S_2$, &c., are functions of $c$, they are constant so long as $c$ is constant, though $a$ may diminish and $x_1$ increase. Thus it appears that $A$ is equal to $a^{1/3}dx$ multiplied by a constant number, and $B \div A$ is equal to $a^{1/3}$ multiplied by another constant number. Assuming any sufficiently large and convenient value of $c$ we can compute $S_1$, $S_2$ &c., and consequently find the numbers in question. With log $c = 2.7$, I found

$S_1 = 19302.99$, $S_2 = 3333.390$, $S_3 = 17550.55$, $S_4 = 3030.792$, and (35) became

$$A = .36a^{1/3}dx, \quad B = -.729012Aa^{1/3}.$$

An identical result had been reached with log $c = 2.6$, except that the last figure of the second coefficient was 1 instead of 2. The first coefficient .36 contains only two figures because $S_1S_4$ and $S_2S_3$ in (35) are so nearly equal that when computed by logarithms to seven figures their difference contains only two figures. The result obtained is accurate enough to show the nature of the curve. We see that $A$ and $B$ vary as $a^{1/3}$ and $a^{2/3}$, respectively. If however $a$ is essentially negative, $S_2$, $S_4$ and $c$ will be also negative, and (35) will give

$$A = -.36a^{1/3}dx.$$

We have then the general values

$$A = \pm.36a^{1/3}dx, \qquad B = \mp.262444a^{2/3}dx, \tag{36}$$

the upper or lower sign being used according as $a$ is essentially positive or negative. Substituting in (34), the equation becomes

$$\begin{aligned} y = &\pm.36a^{1/3}dx[1+H_1(ax^3)+H_2(ax^3)^2+\&c.] \\ &\mp.262444a^{2/3}xdx[1+K_1(ax^3)+K_2(ax^3)^2+\&c.]. \end{aligned} \tag{37}$$

The presence of the factor $dx$ shows that $y$ is an infinitesimal of the first order. Writing $Y = y \div dx$, we have

$$\begin{aligned} Y = &\pm.36a^{1/3}[1+H_1(ax^3)+H_2(ax^3)^2+\&c.] \\ &\mp.262444a^{2/3}x[1+K_1(ax^3)+K_2(ax^3)^2+\&c.], \end{aligned} \tag{38}$$

where the ordinate $Y$ is finite, and is proportional to, but numerically infinitely greater than the elementary area $Ydx = y$ which represents the coefficient of the resultant adjustment formula at the limit. Again, if we write $g = a(dx)^3$, and remember that $x = idx$, (28) and (37) give

$$\begin{aligned} g = \frac{2}{b_3(k+1)}, \quad y = &\pm.36g^{1/3}[1+H_1(gi^3)+H_2(gi^3)^2+\&c.] \\ &\mp.262444g^{2/3}i[1+K_1(gi^3)+K_2(gi^3)^2+\&c.], \end{aligned} \tag{39}$$

which can be used as (32) was, to give approximately the coefficients $l$ when $k$ is a finite number.

We will take, however, $a = .01$, and compute a series of values of $Y$ from (38), and arrange them in Table III. The curve which they follow shows points of inflexion at $x = 0$ and at $y = 0$ as (33) requires.

TABLE III. ($a = \frac{1}{100}$.)

| $x$ | $Y$ | $x$ | $Y$ | $x$ | $Y$ |
|---|---|---|---|---|---|
| —24 | .0591 | —8 | .0826 | 8 | .0115 |
| —22 | .0826 | —6 | .1122 | 10 | .0060 |
| —20 | .0370 | —4 | .1156 | 12 | .0029 |
| —18 | —.0358 | —2 | .1007 | 14 | .0014 |
| —16 | —.0856 | 0 | .0776 | 16 | .0006 |
| —14 | —.0839 | 2 | .0541 | 18 | .0002 |
| —12 | —.0370 | 4 | .0346 | 20 | .0000 |
| —10 | .0278 | 6 | .0206 | 22 | &c. |

For positive values of $x$ in the table, $Y$ is positive and the axis of $X$ appears as an asymptote to the curve. For negative values of $x$, the curve consists of an infinite series of undulations lying alternately above and below the axis, only two of which, however, appear in the table. Since the equation contains only one arbitrary constant $a$, and the area

$$\int_{-\infty}^{\infty} Y dx = \frac{1}{dx}\int_{-\infty}^{\infty} y dx$$

is always unity, the curves for different values of $a$ are similar figures in the same sense as all probability curves are similar. The dimensions in the direction of $x$ vary inversely as those in the direction of $Y$, and since the latter vary as $A$ and therefore as the absolute values of $a^{1/3}$ or $g^{1/3}$, it follows from (28) that the $x$ dimensions vary directly as the cube root of $b_3$ without regard to sign, and also as the cube root of $k+1$. Since the tabulated values of $Y$ are the same as those of $y$ in (39) would be if we took $g = .01$ and gave $i$ the same values which $x$ has in the table, it results that the table can be used to compute approximately the coefficients for a finite number of applications of a given formula. Take for instance eight applications of

$$u'_0 = \tfrac{1}{8}(-u_{-1} + 6u_0 + 3u_1).$$

Here we have $\lambda_0 = \frac{6}{8}$, $\lambda_1 = \frac{3}{8}$, $\lambda_{-1} = -\frac{1}{8}$, $b_0 = 1$, $k = 8$.

The place of the centre of parallel forces is

$$\frac{-1(-1)+6\times0+3\times1}{-1+6+3} = \tfrac{1}{2},$$

or half way from $u_0$ to $u_1$. Taking this centre as an origin, $b_2$ is

$$\tfrac{1}{8}(-\tfrac{9}{4}+\tfrac{6}{4}+\tfrac{3}{4}) = 0, \qquad \text{and } b_3 \text{ is}$$

$$\tfrac{1}{8}(\tfrac{27}{8} - \tfrac{6}{8}+\tfrac{3}{8}) = \tfrac{3}{8},$$

and (39) gives

$$g = \tfrac{16}{27}.$$

To find any resultant coefficient, we take the corresponding $Y$ from the table, and multiply it by the ratio

$$\left(\tfrac{16}{27} \div \tfrac{1}{100}\right)^{\frac{1}{3}} = 3.90.$$

Multiplying this ratio into

$$\ldots . -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ \ldots .$$

we get the abscissas

—23.4, —19.5, —15.6, —11.7, —7.8, —3.9, 0, 3.9, 7.8, 11.7, 15.6, 19.5.

The corresponding ordinates, by interpolation in Table III., are

.075, .018, —.081, —.030, .085, .115, .078, .035, .012, .003, .001, .000.

Multiplying these by 3.90, we have the approximate coefficients of the resultant formula

$$u_0^{\text{viii}} = -.31u_0 - .12u_1 + .33u_2 + .45u_3 + .30u_4 + .14u_5 + .05u_6 + .01 + \&c.$$
$$+ .07u_{-1} + .29u_{-2} - \&c.$$

Since in the original formula the centre of parallel forces lay at an interval of $\frac{1}{2}$ to the right of $u_0$, the resultant coefficient .30 corresponding to $x = 0$ in the table belongs to the term $u_4$, which is at an interval of $k \times \frac{1}{2} = 4$ to the right of $u_0$. The above series of coefficients bears only a general resemblance to that of the true formula for eight applications.

$$u_0^{\text{viii}} = -.11u_0 - .11u_1 + .19u_2 + .42u_3 + .35u_4 + .16u_5 + .04u_6 + \&c.$$
$$+ .04u_{-1} + .02u_{-2} - .02u_{-3} + .01u_{-4} - \&c.,$$

as found by expansion of the polynomial

$$\left(\frac{-1 + 6z + 3z^2}{8}\right)^8,$$

but the approximation would be closer if $k$ were greater. This and other trials which have been made indicate that unsymmetrical limits are not approached as closely as symmetrical ones are, for given finite values of $k$.

When the constant $a$ in (33) is negative, the effect will be to reverse the position of the limiting curve, so that the undulations will be on the right hand of the origin. The sign of $a$ depends upon that of $b_3$, as (28) shows. If the order of the coefficients in the given adjustment formula were reversed, so that for instance in the example just considered, we should have

$$u'_0 = \tfrac{1}{8}(3u_{-1} + 6u_0 - u_1),$$

the resultant coefficients would evidently be the same as before, but in the reverse order. We should still have $b_0 = 1$, and $b_2 = 0$ when referred to the centre of parallel forces, but $b_3$ would become

$$\tfrac{1}{8}\left(-\tfrac{3}{8} + \tfrac{6}{8} - \tfrac{27}{8}\right) = -\tfrac{3}{8},$$

being the same as before only with sign changed. Thus $a$ and $g$ would remain unaltered in absolute value, but with negative signs. The form of the

equation of the limiting curve remains the same, for with given absolute values of $a$ and $x$ in (38), it makes no difference with the value of $Y$ whether we use the upper signs with positive $a$, or the lower signs with negative $a$, provided we reverse the sign of $x$.

Passing now to the third case under (28), where $n=3$, and the equation and its conditions are

$$\frac{d^3y}{dy^3}=axy,\quad \frac{1}{dx}\int_{-\infty}^{\infty} ydx=1,\quad \int_{-\infty}^{\infty} x^2ydx=0,\quad \int_{-\infty}^{\infty} x^3ydx=0, \tag{40}$$

there is not much to be added to what was said in the discussion of symmetrical formulas in my article already referred to. (ANALYST, Sept. 1878, p. 134, and May, 1879, p. 72.) As in the case of $n=1$, so here, the resultant or limiting curve will always be symmetrical with respect to the ordinate through the centre of parallel forces, whether the original coefficients $\lambda$ be so or not. For the $\lambda$ do not enter into the general differential equation (28) of these curves, except as they affect the constant $b_{n+1}$. The constants of even order, $b_2$, $b_4$, $b_6$ &c., are not altered by a reversing of the sequence of the $\lambda$. Take for example the adjustment formula

$$u'_0=\tfrac{1}{81}(-4u_{-1}+30u_0+60u_1-5u_2),$$

and its reverse

$$u'_0=\tfrac{1}{81}(-5u_{-1}+60u_0+30u_1-4u_2).$$

The centre of parallel forces in the first one is $\frac{2}{3}$ of the way from $u_0$ to $u_1$, and consequently in the second one it is $\frac{1}{3}$ of the way from $u_0$ to $u_1$. Since the first one satisfies the three conditions $b_0=1$, $b_2=0$, $b_3=0$, with reference to its centre of forces, the second also satisfies them with reference to its centre of forces, and the value of $b_4$ is in both cases

$$\tfrac{1}{81}[-4(\tfrac{5}{3})^4+30(\tfrac{2}{3})^4+60(\tfrac{1}{3})^4-5(\tfrac{4}{3})^4]=-\tfrac{40}{81},$$

so that $a$ is the same for both. The equations of the two limiting curves will thus be alike in all respects; but one curve is the reverse of the other, therefore the curve must be symmetrical on either side of the axis of $Y$ or ordinate passing through the centre of parallel forces. Its equation is

$$y=.40845a^{1/4}dx\left[1+\frac{ax^4}{2.3.4}+\frac{(ax^4)^2}{2.3.4.6.7.8}+\frac{(ax^4)^3}{2.3.4.6.7.8.10.11.12}+\&c.\right]$$
$$-.1380517a^{3/4}x^2dx\left[1+\frac{ax^4}{4.5.6}+\frac{(ax^4)^2}{4.5.6.8.9.10}+\&c.\right]. \tag{41}$$

We might put this into forms analogous to (38) and (39), taking

$$Y=y\div dx,\qquad x=idx,\qquad g=a(dx)^4.$$

The value of $g$ will be given by (28),

$$g=\frac{-6}{b_4(k+1)}.$$

The significance of $b_4$ here must not be confounded with that which it had in my previous article (Sept. 1878, pp. 134–7), being somewhat different, as already stated in connection with (14) and (21). If $b_4$ is positive, $a$ will be negative, the factors $a^{1/4}$ and $a^{3/4}$ in (41) will be imaginary, and there will be no limiting curve. This corresponds to the case under the probability curve (31), where $\sqrt{(-a)}$, being evidently a factor in the expression for $y$, becomes imaginary when $b_2$ is negative, making $a$ essentially positive. Taking $a = .001$, the values of $Y = y \div dx$ in (41) are given in the subjoined table, for values of $x$ from 0 to 36. They are the same as what were somewhat improperly called values of $y$, at p. 137 of the previous article referred to.

Table IV. ($a = \frac{1}{1000}$.)

| $x$ | $Y$ | $x$ | $Y$ | $x$ | $Y$ | $x$ | $Y$ |
|---|---|---|---|---|---|---|---|
| 0 | .0726 | 10 | .0196 | 20 | —.0068 | 30 | .0010 |
| 1 | .0719 | 11 | .0133 | 21 | —.0059 | 31 | .0010 |
| 2 | .0696 | 12 | .0077 | 22 | —.0048 | 32 | .0009 |
| 3 | .0659 | 13 | .0029 | 23 | —.0037 | 33 | .0008 |
| 4 | .0610 | 14 | —.0009 | 24 | —.0026 | 34 | .0006 |
| 5 | .0550 | 15 | —.0038 | 25 | —.0016 | 35 | .0004 |
| 6 | .0483 | 16 | —.0058 | 26 | —.0007 | 36 | .0002 |
| 7 | .0412 | 17 | —.0070 | 27 | —.0000 | 37 | etc. |
| 8 | .0338 | 18 | —.0075 | 28 | .0005 | 38 | |
| 9 | .0265 | 19 | —.0074 | 29 | .0008 | 39 | |

Now let us take as an illustration the formula just considered,

$$u'_0 = \tfrac{1}{81}(-4u_{-1} + 30u_0 + 60u_1 - 5u_2),$$

for which $b_4 = -\frac{40}{81}$. With six applications we have $k = 6$, and consequently $g = \frac{243}{140}$. To find the resultant coefficients approximately we take the corresponding values of $Y$ from the table and multiply them by the ratio

$$(\tfrac{243}{140} \div \tfrac{1}{1000})^{1/4} = 6.455.$$

First multiplying this ratio into 0, 1, 2, 3 &c., we have the abscissas

0, 6.5, 12.9, 19.4, 25.8, 32.3.

The corresponding ordinates, by interpolation in Table IV, are

.0726, .0448, .0033, —.0072, —.0009, .0009.

Multiplying these by 6.45, we get the approximate coefficients of the resultant formula

$$u_0{}^{\text{vi}} = .468u_4 + .289(u_3 + u_5) + .021(u_2 + u_6) - .046(u_1 + u_7)$$
$$- .006(u_0 + u_8) + .006(u_{-1} + u_9) - \&c.$$

In the original formula the centre of parallel forces was at an interval of $\frac{2}{3}$ to the right of $u_0$, so that the resultant coefficient .468 corresponding to $x = 0$ in the table belongs now to the term $u_4$ which is at the interval $k \times \frac{2}{3} = 4$ to the right of $u_0$. The true resultant formula in this case is

$$u_0^{vi} = .503u_4 + .318u_5 - .020u_6 - .049u_7 + .014u_8 - .002u_9 + \&c.$$
$$+ .259u_3 + .006u_2 - .030u_1 - 002u_0 + .002u_{-1} + \&c.,$$

as found by expansion of

$$\left(\frac{-4+30z+60z^2-5z^3}{81}\right)^6.$$

It does not agree very closely with the other, because $k = 6$ is not a large number.

Reasoning from the analogy of the cases here considered, as well as from the case of $n = 5$, discussed at p. 138 of my former article, we can make some general inferences respecting all the curves included under (28) and (29).

Since they contain only one parameter $a$, and their area is constant, the curves for a given order $n$, are similar figures in the sense already stated. To make this more clear, we may remark that in the probability curve (31), if we write

$$\omega = hx\sqrt{2} = (-a)^{\frac{1}{2}}x,$$

the equation of the curve becomes

$$Y = (-a)^{\frac{1}{2}}\left[(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}w^2}\right],$$

which may be written

$$Y(-a)^{\frac{1}{2}}f_1(\omega).$$

So also in (38), if we take $\omega = a^{\frac{1}{3}}x$, the equation of the curve becomes

$$Y = a^{\frac{1}{3}}\left\{ \pm .36\left(1+\frac{\omega^3}{2.3}+\frac{\omega^6}{2.3.5.6}+\&c.\right)\right.$$
$$\left. \mp .262444\omega\left(1+\frac{\omega^3}{3.4}+\frac{\omega^6}{3.4.6.7}+\&c.\right)\right\},$$

which may be written

$$Y = a^{\frac{1}{3}}f_2(\omega).$$

Again in (41), putting $\omega = a^{\frac{1}{4}}x$, we have the equation

$$Y = a^{\frac{1}{4}}\left\{ .40845\left(1+\frac{\omega^4}{2.3.4}+\frac{\omega^8}{2.3.4.6.7.8}+\&c.\right)\right.$$
$$\left. -.1380517\omega^2\left(1+\frac{\omega^4}{4.5.6}+\frac{\omega^8}{4.5.6.8.9.10}+\&c.,\right)\right\},$$

which we may write

$$Y = a^{\frac{1}{4}}f_3(\omega).$$

Thus it appears that for any curve whose order is $n$ we shall have

$$x = \omega \div (\pm a)^{\frac{1}{n+1}}, \qquad Y = (\pm a)^{\frac{1}{n+1}} f_n(\omega),$$

so that $\omega$ and $f_n(\omega)$ are what $x$ and $Y$ become when $\pm a$ is unity. Regarding $\omega$ as the abscissa and $f_n(\omega)$ as the ordinate of an initial curve, and taking $\omega$ for instance equal to 0, 1, 2, 3, &c., in succession, we can compute the corresponding values of $f_n(\omega)$, and these values will be independent of any particular value which $a$ afterward may receive.

To construct the curve for a given value of $a$, taking the abscissa $\omega$ and ordinate $f_n(\omega)$ of any point in the initial curve, we have simply to divide the one and multiply the other by the same quantity $(\pm a)^{\frac{1}{n+1}}$, in order to get the abscissa $x$ and ordinate $Y$ of a point in the curve required. Likewise if we have given the abscissa $x'$ and ordinate $Y'$ of a curve whose parameter is $a_1$, and wish to find the corresponding abscissa and ordinate of another curve whose parameter is $a_2$, we shall simply have to multiply the $x'$ and divide the $Y'$ by the common ratio

$$(a_1 \div a_2)^{\frac{1}{n+1}},$$

or what comes to the same thing, divide the $x'$ and multiply the $Y'$ by

$$(a_2 \div a_1)^{\frac{1}{n+1}},$$

and this is what we have practically done in various examples. Thus we see that for all this class of curves, the dimensions in the direction of $x$ vary inversely as those in the direction of $Y$, and the $Y$ dimensions vary directly as the $(n+1)$th root of $\pm a$. But by (28), $a$ varies inversely as $b_{n+1}$ and $k+1$, consequently the $x$ dimensions of the curves vary directly as the $(n+1)$th root of $\pm b_{n+1}$ and also as the $(n+1)$ th root of $k+1$. If the order $n$ is odd, the curve will be symmetrical on either side of the axis of $Y$, but if $n$ is even, the curve will be unsymmetrical and its position will be direct or reversed, according to the sign of $b_{n+1}$. Since the equation of the curve always contains the $(n+1)$th root of $\pm a$ as a factor, it may be imaginary if $n$ is odd, but is always real if $n$ is even. This seems to indicate that the absence of any limiting curve can occur only when the order $n$ is odd, so that the curve, if there were one, would be symmetrical.

(*To be concluded in No.* 1, *Vol. VII.*)